Generalized Functions- Exercise 7

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1. (a) We'll show that there exist two coordinate maps $\rho : \mathbb{R}^m \to M$ and $\psi : \mathbb{R}^n \to N$ such that $\psi^{-1} \circ \varphi \circ \rho$ is an injective map. Note that Since any open ball in $\mathbb{R}^k$ is homeomorphic to $\mathbb{R}^k$ itself, we can assume that ρ, ψ are actually maps from U, V open balls to $M.N$ respectively. First we take any two coordinate maps g, h for M, N respectively. Because f is an immersion and g, h are diffeomorphisms we have that $Diff(\psi^{-1} \circ \varphi \circ \rho)$ is injective, so we can assume WLOG that the differential is given by a matrix of the form $\left(\frac{I_m}{0}\right)$ (otherwise we change the basis). Consider then the map $T : U \times \mathbb{R}^{n-m} \to V$ given by $T(z,t) = \psi^{-1} \circ \varphi \circ \rho(z) + (0,t)$, then by construction its differential at $\rho^{-1}(x)$ is I_n, and in particular invertible. Therefore by the inverse function theorem we have thar T is invertible in some neighborhood of $\rho^{-1}(x)$ (in particular injective). We take U, V to be this neighborhood and its image, respectively. But we note that $T(z,0) = \psi^{-1} \circ \varphi \circ \rho(z)$, so $\psi^{-1} \circ \varphi \circ \rho$ is given by a composition of an injective inclusion map $i(z) = (z,0)$ and T, and is thus injective as well.

   (b) We proceed in a completely anologous matter, except that now the differential is of the form $(I_n \mid 0)$ and we take $T : U \to V \times \mathbb{R}^{m-n}$ to be $T(z) = (\psi^{-1} \circ \varphi \circ \rho(z), z_{n+1}, ..., z_m)$ and again we get that the differential of T at $\rho^{-1}(x)$ is I_m, so by the inverse function theorem T is locally invertible and in particular onto. But $\psi^{-1} \circ \varphi \circ \rho(z) = p \circ T(z)$, where $p(z) = (z_1, ..., z_n)$ is the (clearly surjective) projection map, and we are done.

(c) This follows directly from the inverse function theorem.

2. (a) Consider the bilinear map from $C_c^\infty(X) \otimes C_c^\infty(Y)$ to $C_c^\infty(X \times Y)$ sending a pair $f_1 \in C_c^\infty(X)$, $f_2 \in C_c^\infty(Y)$ to the map $f_1 \otimes f_2(x, y) = f_1(x)f_2(y)$ in $C_c^\infty(X \times Y)$ ($f_1 \otimes f_2$ is locally constant, beacause for any point (x, y) f_1 is constant on some open neighborhood $x \in U \subseteq X$, and f_2 is constant open neighborhood $y \in V \subseteq Y$, so $f_1 \otimes f_2$ is constant on $U \times V$, which is open in the product topology.). We claim that this map is an isomorphism. First we show surjectivity: given $f \in C_c'^\infty(X \times Y)$, we claim thatwe can write f as a finite sum $f = \sum c_i 1_{U_i \times V_v}$where U_i, V_i are compact-open sets in X,Y respectively. Indeed, note that for each point (x, y) in $X \times Y$ there exists a neighborhood of the point on which f is constant. Note that since X, Y have bases of open-compact sets, $X \times Y$ has a basis consisting of products of such sets, and therefore there exist U, V open-compact such that $(x, y) \in U \times V$ and f is constant on $U \times V$. Since $supp(f)$ is compact, we can take a finite cover $\{U_i \times V_i\}_{i=1}^{i=N}$ of $supp(f)$. WLOG this cover is disjoint (otherwise we can refine it to make it so). f is identically zero on $(\bigcup U_i \times V_i)^c$, so $f = \sum c_i 1_{U_i \times V_v}$ where c_i is the value of f on $U_i \times V_i$. Now note that $1_{U_i}, 1_{V_i}$are locally constant and $1_{U_i} \otimes 1_{V_i} = 1_{U_i \times V_i}$, so we have surjectivity. Injectivity: suppose that $\sum c_i f_{1.i} \otimes f_{2,i}(x, y) = \sum c_i f_{1,i}(x) f_{2,i}(y) = 0$. Assume that $\{f_{2,i}\}$ aren't all identically zero, and that $\{f_{1,i}\}$ are linearly independent (otherwise we can always pass to a linearly independent subset). Take some $y \in Y$ such that not all $f_{2,i}(y)$ are zero. Then we have for all $x \in X$, $\sum c_i f_{2,i}(y) f_{1,i}(x) = 0$, contradicting linear independence. So $f_{2,i}$ are all identically zero the map is injective.

(b) Take $X = Y = \mathbb{Z}$ (with the discrete topology). These are clearly l-spaces, and we have $C_c^\infty(X) = C_c^\infty(Y)$ are equal to the space of sequences with finite support. So their duals are equal to the space of all real valued sequences. Similarly the dual of $C_c^\infty(X \times Y)$ is the space of all functions from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{R}$ (infinte matrixes). So to prove that this is a counterexample we need to find a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ such that there do not exist $f_{1,i}, f_{2.i} : \mathbb{Z} \to \mathbb{R}$ such that $f(m.n) = \sum c_i f_{1,i}(m) f_{2.i}(n)$ for all m, n. We claim that this is the case for the function $f = 1_{m=n}$ (the infinite dimesional identity matrix). Indeed, soppose that there exist $f_1.f_2$ such that $1_{m=n} = \sum c_i f_{1,i}(m) f_{2,i}(n)$ for all $m.n$. Now note that for any fixed m, we have that $\{f_{2,i}\}$ span the vector $1_{n=m}$. But $\{1_{n=m}\}_{m \in Z}$ is a basis of $\mathbb{R}^{\mathbb{Z}}$, which is an infinite dimesional vector space-contradiction. So $C_c^\infty(X) \otimes C_c^\infty(Y) \neq C_c^\infty(X \times Y)$, as required.

3. (a) We need to show that the map φ^* induced by φ is a homeomorphism of $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$ with itself. By symmetry it suffices to show that the map φ^* is continuous , since its inverse is also induced by a diffeomorphism (φ^{-1}) in the same way. Furthermore, we can assume WLOG that $k = 1$, since for the general case we can work "component by component". Note that φ^* is linear. A linear operator on $C_c(\mathbb{R}^n, \mathbb{R})$ is continuous iff it is sequentially continuous with respect to the following notion of convergence:

$\{f_n\} \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ are said to be convergent to $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ iff

(1)$\{f_n\} \bigcup \{f\}$ are all supported in some compact set $K \subseteq \mathbb{R}^n$and

(2) for every multi index α we have that $\partial^\alpha(f_n)$ tends to $\partial^\alpha(f)$ uniformly.

So it suffices to show that if a sequnce$\{f_n\}$ tends to f (in this sense), then $\{f_n \circ \varphi\}$ tends to $f \circ \varphi$. The first condition is certainly true, because if $\{f_n\} \bigcup \{f\}$ are supported in some compact set K, then $\{f_n \circ \varphi\} \bigcup \{f \circ \varphi\}$ are supported in $\varphi^{-1}(K)$, which is also compact. Proving that the second condition is met is straightforward, but relies on the (rather unpleasant) formula for higher (partial) derivatives of composite functions (the Faà di Bruno formula).

In the one dimensional case the formula reads:

$$(f \circ g)^m(x) = \sum_{\pi \in \Pi_m} f^{(|\pi|)}(g(x)) \cdot \prod_{B \in \pi} g^{(|B|)}(x)$$

where Π_m is the set of all partitions of $\{1, ..., m\}$, $B \in \pi$ if B is one of the blocks of the partition π, $\mid \pi \mid$ denotes the number of blocks of π, and $\mid B \mid$ denotes the size of the block B. We'll also denote the set of all block k-labelings of a partition (i.e functions from the set of blocks to $\{1, ..., k\}$) by π^k. The multivariate version of this formula is then given by

$$\frac{\partial^n}{\partial t_1 \partial t_2 ... \partial t_m}(f \circ g)(x) = \sum_{\pi \in \Pi_m} \sum_{\lambda \in \pi^n} \left\{ \left(\prod_{B \in \pi} \frac{\partial}{\partial \lambda(B)} \right) f(g(x)) \right\} \cdot \left\{ \prod_{B \in \pi} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)}(x) \right] \right\}$$

However, the exact formula isn't important for our purposes. All we need is that when we plug in $f_n \circ \varphi$ and $f \circ \varphi$ into the above, we have uniform convergence for each summand: indeed, the left hand factors converge uniformly (beacuse $\partial^\alpha(f_n)$ tends to $\partial^\alpha(f)$ uniformly for any multi-index) , and the right hand factors are bounded on the compact set $\varphi^{-1}(K)$, which is the only region where f_n can have non vanishing partial derivatives. So we get uniform convergence of each summand, and thus also of the entire sum.

(b) Again, by symmetry it suffices to show the continuity of ψ_*. Also, since $\psi_*(f)(x) = (\sum_{1,j} \psi_{1,j}(x) f_j(x), ..., \sum_{k,j} \psi_{k,j}(x) f_j(x))$, it suffices to show thet for a smooth scalar function $\psi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$, we have that multiplication by ψ induces a continuous function on $C_c^\infty(\mathbb{R}^n, \mathbb{R})$. So let $\{f_n\}$ be a sequence tending to f in $C_c^\infty(\mathbb{R}^n, \mathbb{R})$. We need to show that $\{\psi \cdot f_n\}$ tends to $\psi \cdot f$. Verifying that condition (1) is met is trivial- if $\{f_n\} \bigcup \{f\}$ are supported in a compact set K, then so are $\{\psi \cdot f_n\} \bigcup \{\psi \cdot f\}$. To prove that the second condition is met we use

Leibniz's formula:

$$\partial^\alpha(f * g) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^\beta f) \cdot (\partial^{\alpha-\beta} g)$$

and again we note that we have uniform convergence in each summand, with the left hand factors converging uniformly by assumption and the right hand side ones being bounded in K.

4. Let $f \in C_c^\infty(\mathbb{R}^n)$ and let D be a differential operator. Note that f and all its derivatives vanish outside of $supp(f)$, so $\parallel f \parallel_D = sup_{x\in R^n} \parallel D(f)(x) \parallel = sup_{x\in supp(f)} \parallel D(f)(x) \parallel$ which is a supremum of a continuous function on a compact set, and therefore finite. Conversely, assume that f isn't compactly supported, i.e there exists a sequence of points $x_m \in R^n$ such that $x_m \to \infty$ and $f(x_m) \neq 0$ for all m. WLOG we can assume that $\parallel x_{m+1} - x_m \parallel > 1$ for all m. For each m we can construct a smooth function g_m such that $g_m(x_m) = \frac{m}{f(x_m)}$ and $g_m(x) = 0$ for all x such that $\parallel x - x_m \parallel > 1$. We take $g = \sum g_m$ (this is well defined because at each point at most 1 of the g_m are non-zero), and the operator $D(f) = g \cdot f$. So we have $D(f)(x_m) = m$ and $\parallel f \parallel_D = \infty$, as required.